arXiv:cond-mat/9806301v1   25 Jun 1998

Edge Excitations of an Incompressible Fermionic Liquid in a Disorder Magnetic Field

A. Sedrakyan[1]

ITP, University of Bern, Sidlerstrasse 5, CH-3012, Bern [2]

Abstract

The model of lattice fermions in 2+1 dimensional space is formulated, the critical states of which are lying in the basis of such physical problems, as 3D Ising Model(3DIM) and the edge excitations in the Hall effect. The action for this exitations coincides with the action of so called sign-factor model in 3DIM at one values of its parameters, and represent a model for the edge excitations, which are responsible for the plato transitions in the Hall effect, at other values. The model can be formulated also as a loop gas models in 2D, but unlikely the $O(n)$ models, where the loop fugacity is real, here we have directed (clochwise and conterclochwise) loops and phase factors $e^{\pm 2\pi \frac{p}{q} i}$ for them. The line of phase transitions in the parametric space will be found and corresponding continuum limits of this models will be constructed. It appears, that besides the ordinary critical line, which separates the dense and diluted phases of the models(like in ordinary $O(n)$ models), there is a line, which corresponds to the full covering of the space by curves. The $N = 2$ twisted superconformal models with $SU(2)/U(1)$ coset model coupling constant $k = \frac{q}{p} - 2$ describes this states.

---

[1]e-mail: *sedrak@lx2.yerphi.am; sedrak@nbivax.nbi.dk*

[2]Permanent address: Yerevan Physics Institute, Br.Alikhanian st.2, Yerevan 36, Armenia

1   Introduction

It is well known that in most of important problems of modern physics, such as quark confinement in Quantum Chromodynamics (QCD) or high temperature superconductivity, the understanding and description of ground state is crucial for the construction of the theory of corresponding phenomena. For a systems in the strong coupling regime the ground state is far from trivial vacuum, as we have in the perturbative theories, being essentially structured. Hence, the important step in developing such theories should be the understanding of the ground state.

The aim of this article is the formulation of the lattice model in 2+1 dimensional space and investigation of its critical states, which,as it seems to us, is lying in the basis of such physical problems as $3D$- Ising model ($3DIM$) [1, 2, 3] and edge excitations, responsible for the plato transitions in the Hall effect [4, 6, 7].

In the presented model we have used the essential ingredients of the model for so-called sign-factor, formulated in the string representation if the $3DIM$ [3], namely, the chiral fermions, which are hopping along the arrows on the Manhattan lattice (ML) (Fig.1) in the presence of some external $SU(2)$ gauge field.

The external gauge field in [3] was induced by the random surface in the $3D$ regular lattice and consists on background fluxes per plaquette, which takes values in the Z_2 center of the $SU(2)$ group.

Now, simplifying the situation, we will consider the tight-binding model for electrons, which are hopping along the arrows of ML in the external $U(1)$ field, which formed by $\pm 2\pi p/q$ fluxes per plaquette, arranged in a chess-like order. The model describes the motion of electrons in a regular lattice of magnetic vortices with the staggered directions of fluxes, which are placed in the middle points of plaquettes.

Being interested in a low lying ($E \approx 0$) critical excitations of the model we are changing the creation and annihilation operators of electrons in the proposed Hamiltonian by Grassmann valued fields in two dimensional space and obtain the action for them. By taking functional integral over this Grassmann variables we will find an alternative

formulation for the partition function as loop gas model, expressed as a statistical sum of random non self-intersecting curves (due to ML structure), which appeared with the usual Boltzmann weights as $e^{m_0 L}$ and fugacities equal to phase factor $\eta = -e^{\pm 2\pi \frac{p}{q} i}$. One should not mix this models with the usual loop-gas (O(n) or Q-Potts) models [18, 19], where the fugacities are real and contours are not oriented.

In this article we will consider the limit $\frac{m}{t} \to 0$, where m is the chemical potential and t is the hopping parameters of electrons, when the contours in loop gas formulation of the model fully cover the $2d$ space (or random surface in general case). The limits $m \to 0$ and $t \to \infty$ are equivalent after corresponding rescaling of the partition function and we will refer it as fully packed phase. It will be shown that in this limit the model of low lying states is equivalent to twisted $N = 2$ superconformal field theory [24, 25], the characteristic $SU(2)/U(1)$ coset model coupling constant of which is $k = \frac{q}{p} - 2$. As it is well known [8, 9] this twisted models are topological [10, 11], so we have a statistical physics models corresponding to them.

For sign-factor of the $3DIM$ [3] it is necessary to consider fermions in fully packed phase and in the $\Phi = \pi$-flux background, which means that fugacities of the electronic paths are equal to one. But for the full construction of sign factor it is necessary to add to the action a pair of vertex operators, which corresponds to the creation of additional impurity fluxes $\Phi = \pi$ at some points. This fluxes are induced by the Whitney singularities [12] of the immersed with the defects $2D$ manifolds into the $3D$ Euclidean space, which appeared in sign-factors construction of the $3DIM$ [3]. Here we will not consider impurity fluxes, trying first to construct the continuum limit of the model without them, but proceeding in this way we hope to have a topological description of the sign-factor in terms of the geometry of surfaces.

Another interesting application of our model is Hall effect, which corresponds to another point of parametric space (m, t) The main characteristics of the Hall effect is the fact that the states in the bulk are forming incompressible liquid of localized electrons [5], while there is delocalized states near the edges of the sample, which at the appropriate density of electrons (or at some external magnetic fields) becomes occupied as a Fermi

level and, hence, is responsible for the longitudinal conductivity at the plato transition points. It appears, that our model demonstrates similar future. Moreover, by constructing the transfer matrix for the low lying excitations in our model, one can recognize the transfer matrix, proposed by Chalker and Coddington in [13] for scattering of the edge excitations. It will be shown [14], this long range excitations can be described as a (0,1)-spin fermionic system with the central charge $c = -2$.

In the Section 2 we present the description of the model and calculate the spectrum of excitations.

In the Section 3 we construct the transfer matrix and calculate the spectrum of the model on ML in the absence of the background fluxes. In the appropriate scaling limit of the parameters of the model we will have a massless (0,1)-spin fermionic system with the central charge $c = -2$.

Section 4 intended for the analyze of critical states and on representation of background $U(1)$ fluxes via vertex operators of two fluctuating scalar fields with opposite statistics. The construction of the continuum limit of the whole model for the low lying excitations will be completed in the Section 5.

It is worth to mention now, that proposed in [3] and here Hamiltonian is non-Hermitian. Recently non-Hermitian Hamiltonians have attracted much attention in respect with vortex pinning problem in superconductors [31], directed quantum chaos [32] and in study of sliding of charge density waves in disorder systems [33].

2   Description of the Model

The Manhattan lattice (ML) is the lattice, where there are continuous arrows on the links with the opposite directions on the neighbor parallel lines (Fig.1). The arrows form a set of vectors $\vec{\mu}_{ij} \in \mathcal{S}$. ML originally was defined by Kasteleyn [15] in connection with the problem of single Hamiltonian walk (HW).
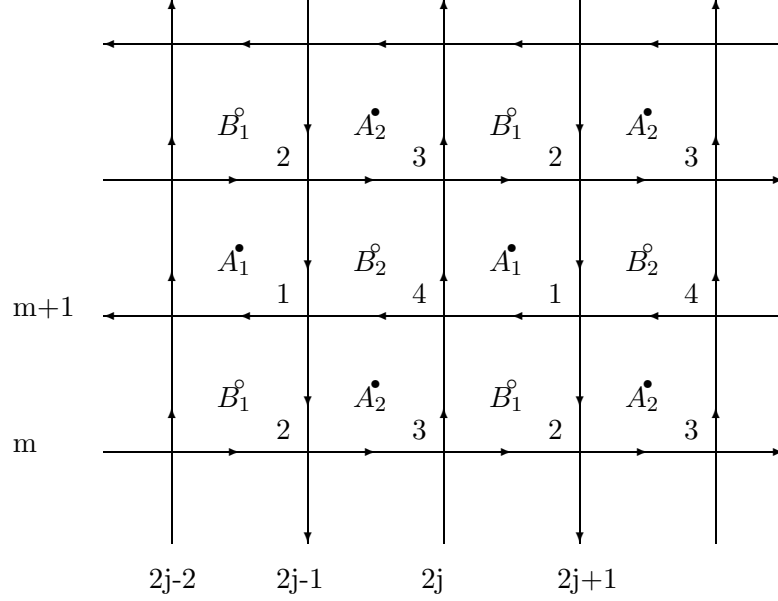
Fig.1.

The multiple HW problem was considered and solved by Duplantie and David [17]. Here we will consider flat ML, though the random scarface with the ML structure, which is necessary to have for $3DIM$, also can be defined easily [3, 19].

The plaquettes of ML are divided into four groups, A_a and B_a (a=1,2), destinated in the chess like order. The A-plaquettes differ from B-plaquettes by the fact, that arrows are circulating around them, while there is no circulation for B-plaquettes. $A_1(A_2)$ has clockwise(counterclockwise) circulation, while B_1 differs from the B_2 by rotation on $\pi/4$ (see fig.1). We will organize the walk of the fermions along the arrows with hopping parameters, which are invariant on double lattice spacings translations, simultaneously being in the disordered $U(1)$ (magnetic) field. The magnetic field is defined as follows. There is $U(1)$ flux $\Phi = 2\pi p/q$ in the A-plaquettes and $\Phi = -2\pi \ p/q$, in the B-plaquettes (p and q are mutually prime integer numbers). On the random ML only the A-type pla-quettes can be deformed into n-angles, while B-type plaquettes will stay as quadrangles.

We will consider one fermionic degree of freedom per lattice site. Because of ML structure the translational invariance occur only for double lattice spacing translations in both directions, the corresponding Brillouin zone of fermions will be reduced and we will

have four bands. Let $C_{\vec{n}}^{i+}, C_{\vec{n}}^{i}$, $(i = 1, 2, 3, 4)$ are corresponding creation and annihilation operators at the lattice sites $\vec{n} = (n, m)$. For example (see Fig.1) we can mark operators as follows

$$
\begin{array}{lll}
C_{\vec{n}}^1 & \text{at the} & \vec{n} = (\text{odd, odd}) \\
C_{\vec{n}}^2 & \text{at the} & \vec{n} = (\text{odd, even}) \\
C_{\vec{n}}^3 & \text{at the} & \vec{n} = (\text{even, even}) \\
C_{\vec{n}}^4 & \text{at the} & \vec{n} = (\text{even, odd})
\end{array}
$$

The Hamiltonian of the model we would like to define is

$$
\begin{aligned}
\mathcal{H}(C_{\vec{n}}^i; \Phi) &= \sum_{\substack{\vec{n}, \vec{\mu} \\ \text{on } ML}} \Big(t_{14} C_{\vec{n}}^{1+} U_{\vec{n}, \vec{\mu}} C_{\vec{n}+\vec{\mu}}^4 + t_{43} C_{\vec{n}}^{4+} U_{\vec{n}, \vec{\mu}} C_{\vec{n}+\vec{\mu}}^3 + t_{32} C_{\vec{n}}^{3+} U_{\vec{n}, \vec{\mu}} C_{\vec{n}+\vec{\mu}}^2 + \\
&+ t_{21} C_{\vec{n}}^{2+} U_{\vec{n}, \vec{\mu}} C_{\vec{n}+\vec{\mu}}^1 + t_{41} C_{\vec{n}}^{4+} U_{\vec{n}, \vec{\mu}} C_{\vec{n}+\vec{\mu}}^1 + t_{34} C_{\vec{n}}^{3+} U_{\vec{n}, \vec{\mu}} C_{\vec{n}+\vec{\mu}}^4 + \\
&+ t_{23} C_{\vec{n}}^{2+} U_{\vec{n}, \vec{\mu}} C_{\vec{n}+\vec{\mu}}^3 + t_{12} C_{\vec{n}}^{1+} U_{\vec{n}, \vec{\mu}} C_{\vec{n}+\vec{\mu}}^2 \Big) + m \sum_{\substack{\text{sites} \\ i=1,..4}} C_{\vec{n}}^{i+} C_{\vec{n}}^i
\end{aligned} \tag{1}
$$

where $\vec{\mu}$ are the unit vectors on ML along of directions of the arrows, m is the chemical potential and relations

$$
U_{\vec{n},\vec{\mu}} \in U(1), \quad \eta_A = \eta = \prod_A UUUU = e^{i\Phi} = e^{2\pi i p/q},
$$

$$
\eta_B = \eta^{-1} = \prod_B UUUU = e^{-i\Phi} \tag{2}
$$

defines the external $U(1)$ magnetic field background. The fermion hopping amplitudes t_{ij} can be regarded as metric elements on the lattice and may be represented as $\rho e^{\Gamma_{\vec{n}, \vec{\mu}}}$, where ρ is the conformal factor and $\ln \rho + \Gamma_{\vec{n}, \vec{\mu}}$ can be regarded as gravitational connection. It is important to mention the absence of the complex conjugate terms in the expression (1) and therefore the Hamiltonian is non-Hermitian.

So we have the Hamiltonian for fermions, which are living on Manhattan lattice in the disordered external magnetic and gravitational backgrounds.

One can write the partition function for the excitations of energy E of the model as

$$
Z(E) = \int \prod_{i=1}^4 \prod_{\vec{n}} dC_{\vec{n}}^i \, d\bar{C}_{\vec{n}}^i \, e^{-\mathcal{H}(C_{\vec{n}}^i; \Phi) + E \sum_{\substack{\text{sites} \\ i=1,..4}} C_{\vec{n}}^{i+} C_{\vec{n}}^i} \tag{3}
$$

where $C_{\vec{n}}^i$ and $\bar{C}_{\vec{n}}^i$ are now independent Grassmann fields, corresponding to $C_{\vec{n}}^i$ and $C_{\vec{n}}^{i+}$.

In the Bloch wave basis and after some simple gauge fixing the Hamiltonian (1) can be written as

$$\mathcal{H}(C_{\vec{n}}^i; \Phi) = \sum_{\vec{k}} C_{\vec{k}}^{i+} \mathcal{H}(\vec{k})_{ij} C_{\vec{k}}^j, \qquad k_x = \frac{2\pi}{L} l, l = 0, ... \frac{L}{2} - 1, \tag{4}$$

$$k_y = \frac{2\pi}{N} n, n = 0, ... \frac{N}{2} - 1,$$

where $\mathcal{H}(\vec{k})_{ij}$ are the elements of the following matrix

$$\mathcal{H}(\vec{k}) = \begin{pmatrix} m & \omega t_{12} e^{-ik_y} & 0 & \omega^{-1} t_{14} e^{-ik_x} \\ \omega^{-1} t_{21} e^{-ik_y} & m & \omega t_{23} e^{ik_x} & 0 \\ 0 & \omega^{-1} t_{32} e^{ik_x} & m & \omega t_{34} e^{ik_y} \\ \omega t_{41} e^{-ik_x} & 0, & \omega^{-1} t_{43} e^{ik_y} & m \end{pmatrix}, \tag{5}$$

and $\omega = e^{\frac{i\pi}{2}\frac{p}{q}}, \omega^4 = \eta$.

It's not hard now to find the spectrum of the model (1) by diagonalizing 4×4 matrix of $\mathcal{H}(\vec{k})$. The answer is obviously invariant under double lattice space periodic $U(1)$ transformations and can be written as

$$E = m \pm \left\{ \frac{1}{2} \left(t_{23} t_{32} e^{2ik_x} + t_{14} t_{41} e^{-2ik_x} + t_{43} t_{34} e^{2ik_y} + t_{12} t_{21} e^{-2ik_y} \right) \pm \right.$$
$$\pm \left[\frac{1}{4} \left(t_{23} t_{32} e^{2ik_x} + t_{14} t_{41} e^{-2ik_x} + t_{43} t_{34} e^{2ik_y} + t_{12} t_{21} e^{-2ik_y} \right)^2 - \right.$$
$$\left. \left. - (t_{14} t_{32} - \eta t_{12} t_{34})(t_{23} t_{41} - \eta^{-1} t_{43} t_{21}) \right]^{1/2} \right\}^{1/2} \tag{6}$$

One can see, that when m=0 we have a $E \to -E$ symmetry in the spectrum of the model, which is consequence of anticommutativity of the Hamiltonian $\mathcal{H}(\vec{k})$ with the chirality operator Γ

$$\left\{ \mathcal{H}(\vec{k}), \Gamma \right\} = 0, \qquad \Gamma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{7}$$

I is a 2×2 unit matrix.

Let as now to investigate the critical $E \approx 0$ states of the system and consider homogeneous $t_{ij} = t$ case for simplicity. It is easy to find from (6), that $E = 0$ for

$$\cos 2k_x + \cos 2k_y = \frac{m^2}{2t^2} + 2(\sin \frac{\Phi}{2})^2 \frac{t^2}{m^2}, \tag{8}$$

6

and the only solutions of this equation in the limit $\frac{m}{t} \to 0$ are imaginary k_x (or k_y) with $|k| \to \infty$. In general, the real solutions appears by appropriate choose of hopping parameters t_{ij} both, for finite and infinite limits, only for $\Phi = 0$. This means that wave functions of $E = 0$ states exponentially grows to the boundary of the sample, localizing fermions there. One can call this states as edge states. What we will do further in this article, is just to prove, that in a case of appropriate choose of hopping parameters, this edge excitations are massless and find the corresponding field theory for them.

The partition function for this edge excitations is simply $Z(0)$ in (3), which means that the original Hamiltonian (1) in 2+1 space, written in Grassmann fields (instead of creation-annihilation operators), can be considered as the action for them in $2D$.

$$A(C_{\vec{n}}^i; \Phi) = \mathcal{H}(C_{\vec{n}}^i; \Phi) \tag{9}$$

Without loss of generality we can re-scale Grassmann fields in the action $A(C_{\vec{n}}^i; \Phi)$ in order to fix $m = 1$. This will require an appropriate rescaling of the partition function.

Because of Grassmann properties of the fields $C_{\vec{n}}^i$, $Z(0)$ is equal to the sum of products of Wilson loop integrals from $U(1)$ and gravitational fields over all possible oriented contours, which covers the lattice. The $U(1)$ loop integrals represents the magnetic fluxes through the contours, which is always equal to $\Phi = \pm 2\pi i p/q$, due to the disordered character and the fact, that in ML all closed curves contain odd number of plaquettes.

Therefore

$$Z(0) = \int \prod_{i=1}^{4} \prod_{\vec{n}} dC_{\vec{n}}^i \, d\bar{C}_{\vec{n}}^i \, e^{-A(C_{\vec{n}}^i; \Phi)} = \sum_{\substack{\text{all possible} \\ \text{coverings}}} (-\eta)^{N_1} (-\eta)^{-N_2} \prod_{\sigma, \tau} \Phi(\gamma_\sigma) \Phi(\bar{\gamma}_\tau), \tag{10}$$

where $N_1(N_2)$ is the number of the closed contours $\gamma_\sigma(\bar{\gamma}_\tau), (\sigma, \tau = 1....N_1(N_2))$ with clockwise (anti-clockwise) orientations and

$$\Phi(\gamma) = \prod_{\vec{n} \in \gamma} \rho e^{\Gamma_{\vec{n}, \vec{\mu}}}. \tag{11}$$

The sum in (10) is taken over all possible coverings. We formally will not make difference between clockwise and counterclockwise notations for loops in the future if its not necessary.

In the homogeneous case, when $t_{ij} = t$, one can find a similarity of (10) with the critical Q-state Potts or the $O(n)$ loop gas models, when the loop weight factor(the fugacity)

$$- \eta = \sqrt{Q} = n, \tag{12}$$

(see [16, 18, 19]). The essential difference of our model from the mentioned ones is the fact, that instead of real loop weight factor we have a phase factor $-\eta$, defined by (2).

After corresponding rescaling of $Z(0)$ in the limit $\frac{m}{t} \to 0$ by t^{-LN} (LN is the number of ML lattice sites) only the dense coverings of the lattice by loops will survive (the mass term will not contribute) and all loops will appear only with fugacities $-\eta^{\pm 1}$. The $p/q = 1/2$ case is distinguished by its connection with the sign factor of $3DIM$ [3]. For this case we have $-\eta = 1$. The generalization of the model for random surfaces with ML structure on them is straightforward and we will discuss it later. The sign of the surfaces, appeared here, are essentially connected with the singularities at the endpoints of selfintersection lines of the immersed surface [3]. They can be regarded as $U(1)$ flux impurities, which ensure phase factor (-1) for the each contour rounding them.

What immediately follows from (8) is that at the

$$(\frac{m}{t})^2 = 2 \pm \sqrt{4 - (2\sin \Phi/2)^2} = 2 \pm \sqrt{2 + 2\cos \Phi} \tag{13}$$

we have critical behavior of the free energy of $2D$ theory and long range excitations in the spectrum. Formula (13) very much reminds the well known formula for $2D$ $O(n)$-loop gas models [16, 18, 19] phase transition condition [16], if

$$n = -2\cos \Phi$$

What we are going to demonstrate here is that the model has continuum limit (transition line) for all p/q at $t \to \infty$ too.

3 The Spectrum of the Model in the absence of External Magnetic Field

As we can see from the expression (8), in the limit of $t \to \infty$, which needs for a fully packed phase, there is obvious massless excitations in the limit $\Phi \to 0$. This fact determines the way to proceed further, namely we will consider fermionic fields in the empty ($\Phi \to 0$) background and construct his continuum limit. The external $U(1)$ fields and background fluxes will be included into the model via correlators of two scalar fields, φ_1 and φ_2, which interacts with fermions.

In this section we will analyze the model in case of $\Phi \to 0$. It is appeared that the best approach of handling the problem is the Transfer Matrix approach. Following [20] let's find quantum Hamiltonian on a chain, such that his coherent-state path-integral along imaginary time produces action, which coincides with $A(\psi^i; 0)$

$$Z(\Phi = 0) = Tr T^{N/2} = \int \prod_{j,m} d\bar{\psi}_{j,m} d\psi_{j,m} e^{-A(\psi^i;0)} \tag{14}$$

From the Fig.1 we see, that due to geometric properties of ML, the translational invariance of our model is broken for one-lattice spacing translations and present only for two-lattice spacing translations. That means the Transfer Matrix we need is the product of two different Transfer Matrices, defined on two, B_1 and B_2, horizontal chains (choose vertical direction as imaginary time).

$$T = T_{B_1} T_{B_2} =: e^{-H_{B_1}} :: e^{-H_{B_2}} :, \tag{15}$$

where :: means some ordering of operators and will be defined below. The Transfer Matrix will act on a Hilbert space of fermionic states on a 1D chain.

The geometry of ML shows (see Fig.1) that one can define T_{B_1} on a chain of B_1 plaquettes (and correspondingly T_{B_2}- on a B_2) and that is enough to restore the action (1). The structure of Transfer Matrices is easy to fix by shrinking vertical lines (time direction) to points on a chain. Lets define H_{B_1} and H_{B_2} as follows

$$-H_{B_1} = \sum_j \left(t_{23} c_{2j+1}^+ c_{2j} - t_{41} c_{2j}^+ c_{2j+1} + M_1 c_{2j}^+ c_{2j} + M_1' c_{2j+1}^+ c_{2j+1} \right),$$

$$-H_{B_2} = \sum_j \left(-t_{32} c^+_{2j} c_{2j-1} + t_{14} c^+_{2j-1} c_{2j} + M_2 c^+_{2j} c_{2j} + M'_2 c^+_{2j+1} c_{2j+1} \right). \qquad (16)$$

Here c^+_i and c_i are usual fermion creation-annihilation operators, defined on a vacuum $|0>$. The hopping parameter t_{ij} are the same as before, while $M_{1,2}$ and $M'_{1,2}$ have to be specified later.

Let's define coherent states on the sites of chain as follows:

$$|\psi_{2j}\rangle = e^{\psi_{2j} c^+_{2j}}|0\rangle,$$
$$\langle \bar{\psi}_{2j}| = \langle 0| e^{c_{2j} \bar{\psi}_{2j}} \qquad (17)$$

for the even sites and

$$|\bar{\psi}_{2j+1}\rangle = (c^+_{2j+1} - \bar{\psi}_{2j+1})|0\rangle,$$
$$\langle \psi_{2j+1}| = \langle 0|(c_{2j+1} + \psi_{2j+1}) \qquad (18)$$

for the odd sites. The Grassmann variables $\bar{\psi}_i, \psi_i$ anti-commutes with operators c^+_i, c_i.

This states has following properties

$$c_{2j}|\psi_{2j}\rangle = -\psi_{2j}|\psi_{2j}\rangle,$$
$$\langle \bar{\psi}_{2j}|c^+_{2j} = -\langle \bar{\psi}_{2j}|\bar{\psi}_{2j},$$
$$c^+_{2j+1}|\bar{\psi}_{2j+1}\rangle = \bar{\psi}_{2j+1}|\bar{\psi}_{2j+1}\rangle, \qquad (19)$$
$$\langle \psi_{2j+1}|c_{2j+1} = -\langle \psi_{2j+1}|\psi_{2j+1},$$

with the overlap of two different type of coherent states

$$\langle \bar{\psi}_{2j}|\psi_{2j}\rangle = e^{\bar{\psi}_{2j} \psi_{2j}},$$
$$\langle \psi_{2j+1}|\bar{\psi}_{2j+1}\rangle = e^{\bar{\psi}_{2j+1} \psi_{2j+1}}, \qquad (20)$$

and the completeness relation of the states as

$$\int d\bar{\psi}_{2j} d\psi_{2j} |\psi_{2j}\rangle\langle \bar{\psi}_{2j}| e^{\psi_{2j} \bar{\psi}_{2j}} = 1,$$
$$\int d\bar{\psi}_{2j+1} d\psi_{2j+1} |\bar{\psi}_{2j+1}\rangle\langle \psi_{2j+1}| e^{\psi_{2j+1} \bar{\psi}_{2j+1}} = 1. \qquad (21)$$

The state of the chain at the time m is product of the states at sites

$$
\begin{aligned}
|\psi(m)\rangle &= \prod_{j=1}^{L/2} |\psi_{2j}(m)\rangle|\bar{\psi}_{2j+1}(m)\rangle, \\
\langle\psi(m)| &= \prod_{j=1}^{L/2} \langle\bar{\psi}_{2j}(m)|\langle\psi_{2j+1}(m)|,
\end{aligned}
\tag{22}
$$

where L is the length of the chain.

Now the calculation of partition function with Transfer Matrix T by use of formulas (15-22) is easy to perform. If we will choose usual normal ordering for the fermions at even points of the chain and the hole operator ordering at odd points of the chain in the expression (15) for T, then the matrix elements between coherent states, defined in (17-18), can be obtained simply by changing operators c_i, c_i^+ by their eigenvalues $\psi_i, \bar{\psi}_i$. This type of ordering means that in ground state half of states are filled by fermions. Then for the partition function we will have

$$
\begin{aligned}
Z_0 &= TrT^{N/2} = Tr(T_{B_1}T_{B_2})^{N/2} = \\
&= \int \prod_{j,m} d\bar{\psi}_{j,m}d\psi_{j,m}e^{-S(\bar{\psi},\psi)}
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
S(\bar{\psi},\psi) &= \sum_{j,m}\Big[t_{23}\bar{\psi}_{2j+1}(m)\psi_{2j}(m) + t_{41}\bar{\psi}_{2j+1}(m+1)\psi_{2j}(m+1)+ \\
&+ t_{32}\bar{\psi}_{2j+2}(m+2)\psi_{2j+1}(m+2) + t_{14}\bar{\psi}_{2j+1}(m+1)\psi_{2j+2}(m+1)+ \\
&+ (1+M_1)\bar{\psi}_{2j}(m)\psi_{2j}(m+1) + (1+M_2)\bar{\psi}_{2j}(m+1)\psi_{2j}(m+2)+ \\
&+ (1-M_1')\bar{\psi}_{2j+1}(m)\psi_{2j+1}(m+1) + (1-M_2')\bar{\psi}_{2j+1}(m+1)\psi_{2j+1}(m+2)\Big] + \\
&+ \sum_{j,m}\Big[\bar{\psi}_{2j+1}(m)\psi_{2j+1}(m) + \bar{\psi}_{2j}(m)\psi_{2j}(m)\Big],
\end{aligned}
\tag{24}
$$

which is exactly coinciding with the expression (1) for zero external magnetic field, if parameters t_{ij} relates to $M_{1,2}$ and $M_{1,2}'$ as follows

$$
\begin{aligned}
t_{43} &= 1+M_1; & t_{34} &= 1+M_2, \\
t_{21} &= 1-M_1'; & t_{12} &= 1-M_2'.
\end{aligned}
\tag{25}
$$

Our aim is the investigation of the Transfer Matrix T in the limit $t \to \infty$ after the rescaling of partition function by t^{-V}. This is necessary for 3DIM sign factor.

For construction of the continuum limit of the Hamiltonian H of the model defined by

$$T = e^{-H} =: e^{-H_{B_1}} :: e^{-H_{B_2}} :, \qquad (26)$$

we need to analyze his spectrum. Let make Furies transformation of the fermionic fields c_{2i} and c_{2i+1} in (26) separately

$$
\begin{aligned}
c_{2j+1} &= \frac{1}{\sqrt{L}} \sum_p c_{1,p} e^{i(2j+1)p} \\
c_{2j} &= \frac{1}{\sqrt{L}} \sum_p c_{2,p} e^{i2jp}
\end{aligned}
\qquad (27)
$$

where $p = \frac{2\pi}{L} l$, $l = 0, ...L/2 - 1$ are the momentums, which corresponds to Bloch states in the problem with the translational invariance on two lattice spacing. We have two bands in the reduced Brillouin zone. It is obvious that the full Transfer Matrix T can be represented as

$$T = \prod_p : T_{1p} :: T_{2p} := \prod_p T_p, \qquad (28)$$

where T_{1p} and T_{2p} are the Furies transforms of $e^{-H_{B_1}}$ and $e^{-H_{B_2}}$ correspondingly.

Our aim is now to represent the product of normal ordered forms of T_{1p} and T_{2p} in the expression (28) for T_p as

$$T_p = \mathcal{D} e^{\vec{\varepsilon}_p \vec{S} + \mu(n_{1,p} + n_{2,p})} = \mathcal{D} e^{-H_p} \qquad (29)$$

Here the operators

$$S^+ = c_{2,p}^+ c_{1,p}; \quad S^- = c_{1,p}^+ c_{2,p}; \quad S^3 = c_{2,p}^+ c_{2,p} - c_{1,p}^+ c_{1,p} \qquad (30)$$

form an Sl_2 algebra and can be represented as Pauli matrices, while

$$n_{i,p} = c_{i,p}^+ c_{i,p}, \quad \imath = 1, 2 \qquad (31)$$

are particle number operators.

After some obvious, but not trivial calculations we will have for $\varepsilon_p = (\bar{\varepsilon}_p^2)^{1/2}$ the chemical potential μ and $\mathcal{D}$ following expressions

$$
\begin{aligned}
\mathcal{D} &= 2t_{12}t_{21} \\
e^{2\mu} &= \frac{t_{34}t_{43}}{t_{12}t_{21}} \\
\cosh \varepsilon_p &= \frac{1 + \Delta_1\Delta_2 + t_{23}t_{32}e^{2ip} + t_{14}t_{41}e^{-2ip}}{2(t_{12}t_{21}t_{34}t_{43})^{1/2}}.
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
\Delta_1 &= t_{21}t_{43} - t_{41}t_{23}, \\
\Delta_2 &= t_{12}t_{34} - t_{14}t_{32}.
\end{aligned}
\tag{33}
$$

In the Appendix we will represent an alternative way of obtaining of this formulas.

Diagonalization of the Hamiltonian H_p in (29) is easy to realize considering it as an Sl_2 algebra element. It is clear that

$$
H_d = -\varepsilon_p S^3 - \mu(n_{1,p} + n_{2,p}),
\tag{34}
$$

hence the dispersion relations for two bands of the spectrum are

$$
\begin{aligned}
E_{1,p} &= (\varepsilon_p - \mu), \\
E_{2,p} &= (-\varepsilon_p - \mu).
\end{aligned}
\tag{35}
$$

Though we start chapter by talking about absence of the external $U(1)$ field but it is easy to follow that our expressions remain true for arbitrary double lattice spacing periodic external fields. In this case simply the hopping parameters t_{ij} are complex and the phases represent the $U(1)$ field. In the gauge used in (5) we will have the same expression (32) for the spectrum, but where now

$$
\begin{aligned}
\Delta_1 &= \omega^{-2}t_{21}t_{43} - \omega^2 t_{41}t_{23}, \\
\Delta_2 &= \omega^2 t_{12}t_{34} - \omega^{-2}t_{14}t_{32}.
\end{aligned}
\tag{36}
$$

The expression (13) for the critical line follows from the (32) and (36) as a condition $\varepsilon_p = 0$.

Up to now we were considering the spectrum in general case, but for our loop gas model in a fully packed regime it is necessary to consider homogeneous $t_{ij} \to t \to \infty$ limit. As immediately follows from the (32), this limit is singular and needs to be analyzed properly.

Let as go to the limit $t \to \infty$ together with $\Phi \to 0$ always staying on the critical line (13), that is

$$t_{23} = t_{32} = t_{14} = t_{41} = t_{21} = t_{12} = t_{34} = t_{43} = t = \frac{1}{2 \sin \frac{\Phi}{4}}. \tag{37}$$

Then, it easy to see from (32), that in the limit $\Phi \to 0$ we will have $\mu = 0$ and

$$\cosh \varepsilon_p = 2 + \cos 2p. \tag{38}$$

In the next section we will argue why this particular scaling limit was chosen.

From the (35) one can see that the spectrum of the model is gap-less and at $p = \pm \frac{\pi}{2}$ two bands touch each other with $\varepsilon_{\pm \frac{\pi}{2}} = 0$. Therefore, when the ground state is filled up to Fermi level corresponding to $\pm \pi/2$, the spectrum is linear near that point

$$\varepsilon_p = \pm 2k, \tag{39}$$

and we can construct $SO(2)$ invariant field theory. This corresponds to half filling of the states. The situation is similar to one in the Luttinger liquid [22], but it is necessary to calculate the central charge of this massless excitations.

The spectrum was analyzed in [17] in connection with Hamiltonian walk problem by calculation of the partition function and there was established, that it corresponds to $(0,1)$ spin fermionic system with central charge $c = -2$. We will confirm now this result by calculating of the finite size corrections to ground state energy of the model [30].

Following [17] we use the Euler-MacLaurin formula for the approximations of finite sums

$$\begin{aligned} E_0 &= \sum_p \varepsilon_p = L/2\pi \int_0^\pi \varepsilon_p dp + \\ &+ \frac{2\pi}{12L} \left(\varepsilon'(\pi/2) - \varepsilon'(-\pi/2) \right) + ..., \end{aligned} \tag{40}$$

14

where ε_p is the negative energy branch of the spectrum and $\varepsilon'(\pi/2)$ and $\varepsilon'(-\pi/2)$ are the velocities of right(left) excitations at the Fermi level, equal to

$$\varepsilon'(\pi/2) = -\varepsilon'(-\pi/2) = 2. \tag{41}$$

Comparing (40) with the finite size correction formula of excitations with central charge c

$$E = -L\varepsilon_0 - \frac{\pi v}{6L}c, \tag{42}$$

one will obtain $c = -2$ and hence, we have a fermionic system of spins (0,1) [23].

It is known, that (0,1) spin system is just the Faddeev-Popov ghost fields for area preserving transformations.

As it was stressed before, for the partition function of the sign-factors model (let us mention here again that for the full sign-factor we should add to model additional vertex operators as perturbations, which corresponds to Whitney singularities) we should absorb t into $\psi, \bar{\psi}$ by the scale transformations

$$\psi' = t^{1/2}\psi, \quad \bar{\psi}' = t^{1/2}\bar{\psi} \tag{43}$$

and appropriately rescale the partition function $Z(\Phi \to 0)$ by the t^{LN} due to fermionic measure in (14), where LN is the number of fermionic degrees of freedom. Then we will come to

$$\frac{1}{t^{LN}}Z(\Phi \to 0) = \frac{\mathcal{D}^{\frac{LN}{4}}}{t^{LN}}Z(c = -2) = (4\sin\frac{\Phi}{4})^{\frac{LN}{4}}Z(c = -2). \tag{44}$$

Analyzing the $Det\mathcal{H}(\vec{k})$ in a fully packed phase $m \to 0$ one can easily see, that $(4\sin\frac{\Phi}{4})^{\frac{LN}{4}}$ behavior (which $\to 0$ with $\Phi \to 0$) appeared in the answer as a result of phase factor interference in the sum over all possible coverings of the surface by loops, while any particular representative of this sum is finit in the limit $\Phi \to 0$. The ratio of $Det\mathcal{H}(\vec{k})$ over $(4\sin\frac{\Phi}{4})^{\frac{LN}{4}}$, which is $Z(c = -2)$, represents in the limit of the fully packed phase the sum over the variety of coverings of space by curves with the fixed number of loops.

It is worth to mention in the end of this section, that in case of following choose of parameters

$$t_{41} = t_{12} = t_{23} = -t_{34} = \tanh\theta,$$

$$t_{14} = t_{21} = t_{32} = -t_{43} = \frac{1}{\cosh\theta}, \tag{45}$$

the transfer matrix constructed here in (28-33) will be field theory realization of Chalker and Coddingtons transfer matrix for the scattering edge excitations in the Hall effect. They again form $(0,1)$-spin system with the central charge $c = -2$ [14].

4 Interaction with the External Magnetic Field

In this section we will reproduce external $U(1)$ field and background fluxes via two scalar fields.

First, let us introduce dual Manhattan Lattice $\widetilde{ML}$ with sites in the middle points of the ML plaquettes (A and B points in Fig.1) and arrow structure with vectors $\vec{\nu}$ on the links. Vectors $\vec{\mu}$ and $\vec{\nu}$ on ML and $\widetilde{ML}$ are dual each other and intersects perpendicularly in the middle points of the corresponding links.

Consider scalar fields $\varphi(\vec{n})$ and $\bar{\varphi}(\vec{n'})$ $(\vec{n} \in ML, \vec{n'} \in \widetilde{ML})$ on the ML and $\widetilde{ML}$ correspondingly and define following action for them

$$S(\varphi, \bar{\varphi}) = \frac{1}{\pi} \sum_{\substack{\vec{n}, \vec{n'} \\ \vec{\mu}, \vec{\nu}}} \left(\varphi(\vec{n} + \vec{\mu}) - \varphi(\vec{n})\right) \left(\bar{\varphi}(\vec{n'} + \vec{\nu}) - \bar{\varphi}(\vec{n'})\right). \tag{46}$$

In the (46) $\vec{n}$ and $\vec{n'}$ are neighbor sites on ML and $\widetilde{ML}$ correspondingly, vectors $\vec{\mu}$ and $\vec{\nu}$ exits from $\vec{n}$ and $\vec{n'}$ and crosses each other in the middle points of the links.

By introducing $\varphi_1(\vec{n})$ and $\varphi_2(\vec{n})$ as

$$\varphi_1(\vec{n}) = \frac{1}{2}\left(\varphi(\vec{n}) + \bar{\varphi}(\vec{n})\right), \quad \bar{\varphi}_2(\vec{n}) = \frac{1}{2}\left(\varphi(\vec{n}) - \bar{\varphi}(\vec{n})\right) \tag{47}$$

we can separate the holomorphic $\varphi_{1,2}(z), (z = n + im)$ and antiholomorphic $\varphi_{1,2}(\bar{z}), (\bar{z} = n - im)$ parts in each of the fields, which will have following correlators [21]

$$\langle \varphi_1(z)\varphi_1(w) \rangle = -G(|z - w|) - i \, arg(z - w),$$

16

$$\langle \varphi_2(z)\varphi_2(w) \rangle \;\; = \;\; +G(|z-w|) + i\,arg(z-w) \tag{48}$$

In the continuum limit correlators transforming into ordinary correlators of free scalar fields in 2D.

$$G(|z-w|)_{\epsilon \to 0} \to \ln|z-w|, \tag{49}$$

and hence

$$G(z-w)_{\epsilon \to 0} \to \ln(z-w). \tag{50}$$

Let us introduce now following vertex operators

$$
\begin{aligned}
V_A(z,\bar{z}) &= e^{i\frac{Q_1}{4\pi}\Omega\varphi_1 + i\frac{Q_2}{4\pi}\Omega\varphi_2}, \\
V_B(z,\bar{z}) &= e^{-i\frac{Q_1}{4\pi}\Omega\varphi_1 - i\frac{Q_2}{4\pi}\Omega\varphi_2},
\end{aligned}
\tag{51}
$$

The charges $Q_{1,2}$ in (51) have to be defined whereas $\Omega(w) = R(w)\epsilon^2$ (ϵ^2 is the lattice spacing) can be interpreted as solid angle of the plaquette, induced by background curvature $R_b(w)$. We define the following background curvature in this case

$$
\Omega_0(w) = \begin{cases} +2\pi & w \in A \\ -2\pi & w \in B \end{cases}
\tag{52}
$$

Let us define now a new fermionic fields in the sites z of the original ML lattice

$$
\begin{aligned}
\psi_1(z) &= e^{i\frac{Q_1}{2}\varphi_1(z) + i\frac{Q_2}{2}\varphi_2(\bar{z})} C_1(z,\bar{z}) = V_1(z,\bar{z})C_1(z,\bar{z}), \\
\psi_2(z) &= e^{-i\frac{Q_1}{2}\varphi_1(z) - i\frac{Q_2}{2}\varphi_2(\bar{z})} C_2(z,\bar{z}) = V_2(z,\bar{z})C_2(z,\bar{z}), \\
\psi_3(z) &= e^{i\frac{Q_1}{2}\varphi_1(z) + i\frac{Q_2}{2}\varphi_2(\bar{z})} C_3(z,\bar{z}) = V_3(z,\bar{z})C_3(z,\bar{z}), \\
\psi_4(z) &= e^{i\frac{Q_1}{2}\varphi_1(z) - i\frac{Q_2}{2}\varphi_2(\bar{z})} C_4(z,\bar{z}) = V_4(z,\bar{z})C_4(z,\bar{z}).
\end{aligned}
\tag{53}
$$

In the expression (53) $\varphi_1(z)$ and $\varphi_2(\bar{z})$ are the holomorphic and antiholomorphic parts of the scalar fields φ_1 and φ_2 correspondingly. The $\bar{\psi}_i(z)$ fields are defined by complex conjugation. The necessity of the introduction of the two φ_1 and φ_2 fields is dictated by the number of degrees of freedom of original $C_i(z)$ fermionic fields. Also, as we will see later, the presence of two scalar fields $\varphi_{1,2}$ with opposite sign of kinetic energy and appropriate choose of $Q_{1,2}$, insures the cancellation of total central charge of the theory,

which we need in the model (1, 10) by definition. The choice of coefficients $\frac{Q_{1,2}}{2}$ in the exponents of (53) is dictated by coincidence of parameters of the Jacobian of $C_i \to \psi_i$ transformations with the background charges $\frac{Q_{1,2}}{4\pi}\Omega$ in V_A, V_B (see (51)).

We would like to show, that the partition function (10) is equal to

$$Z(0) = \int \prod_{i=1}^{4}\prod_{\vec{n}} d\bar{C}_{\vec{n}}^i dC_{\vec{n}}^i \prod_{\substack{z \in \text{sites} \\ \alpha=1,2}} d\varphi_\alpha(z)d\varphi_\alpha(\bar{z}) \prod_{z \in \text{plaquettes}} V_A(z,\bar{z})V_B(z,\bar{z})e^{S(\varphi_1,\varphi_2)+A(\psi_i;0)}$$

(54)

where $S(\varphi_1\varphi_2)$ is the action of the free scalar fields φ_1, φ_2 with correlators (48) and $A(\psi_i;0)$ is the fermionic action (9) on ML, but with fields (53), placed in the zero external magnetic field.

Let's first make integration over $C_{\vec{n}}^i$. Because of Grassmann character of variables $C_{\vec{n}}^i$, the nonzero contribution in integral (54) can have only terms, where at each site z of ML we have one pair $\bar{C}_{\vec{n}}^i C_{\vec{n}}^i$. The expansion of $e^{A(\psi_i;0)}$ in (54) over ψ_i produces two type of nonzero products of $\bar{\psi}_i(z)\psi_i(z)$ at the each site of ML. The first come as a product of hopping terms in $A(\psi_i;0)$ along closed oriented contours $\gamma_\sigma, (\sigma = 1, ..., k)$ on ML. The second come as a product of mass terms $M\bar{\psi}_i(z)\psi_i(z)$ in $A(\psi_i;0)$ along another closed oriented contours $\bar{\gamma}_\tau, (\tau = 1, ..., l)$, which covers the complement to γ_σ sites. The sum of contours $\gamma_\sigma U \bar{\gamma}_\tau$ densely covers ML by the all possible ways. Hence the term, which have nonzero contribution into partition function $Z(0)$ is the following

$$\mathcal{P} = \sum_{\substack{\text{all possible} \\ \text{coverings}}} \prod_{\sigma=1}^{k} \prod_{\vec{n} \in \gamma_\sigma} \rho e^{\Gamma_{\vec{n},\vec{\mu}}} \prod_{z \in \gamma_\sigma} \bar{\psi}_i(z)\psi_j(z+\mu) \prod_{\tau=1}^{l} \prod_{z \in \bar{\gamma}_\tau} \bar{\psi}_i(z)\psi_i(z) =$$

$$\sum_{\substack{\text{all possible} \\ \text{coverings}}} \prod_{\sigma=1}^{k} \prod_{\vec{n} \in \gamma_\sigma} \rho e^{\Gamma_{\vec{n},\vec{\mu}}} \prod_{z \in \gamma_\sigma} \bar{C}_i(z)C_j(z+\mu)V_i^+(z)V_j(z+\mu) \prod_{\tau=1}^{l} \prod_{z \in \bar{\gamma}_\tau} \bar{C}_i(z)C_i(z) \quad (55)$$

The integration over φ_α in (54) means the calculation of average of the product of operators V_A, V_B and φ_α dependent parts of the fields $\psi_i(z)$ in (55) (see definition of φ_i in (53)).

Therefore

$$Z(0) = \int \prod_{i=1}^{4}\prod_{\vec{n}} dC_{\vec{n}}^i d\bar{C}_{\vec{n}}^i \tilde{Z},$$

$$\tilde{Z} \;=\; \langle \prod_{z\in \text{ plaquettes}} V_A(z,\bar{z})V_B(z,\bar{z})\mathcal{P}\rangle_{\varphi_1\varphi_2} Z_\varphi = Z_1 Z_2 Z_3 Z_\varphi \tag{56}$$

where Z_φ is the partition function of the free scalar fields $\varphi_{1,2}$

$$Z_\varphi = \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 e^{-S(\varphi_1,\varphi_2)} \tag{57}$$

The variable Z_1 in (56) consists only of mutual correlators of vertex operators $V_{A,B}$ and equal

$$Z_1 = \exp\left(\sum_{\substack{z,w\in \text{ plaquettes} \\ \text{of } ML}} \left(\left(\frac{Q_1}{4\pi}\right)^2 - \left(\frac{Q_2}{4\pi}\right)^2 \right) \Omega(z)G(|z-w|)\Omega(w) \right) \tag{58}$$

The Z_2 consists of mutual correlators of $\varphi_{1,2}$, dependent parts of Grassmann fields ψ_i in (58) and (56).

$$Z_2 = \exp\left(\sum_{\substack{z,w\in \text{ sites} \\ \text{of } ML}} \left(\left(\frac{Q_1}{2}\right)^2 - \left(\frac{Q_2}{2}\right)^2 \right) M(z)G(|z-w|)M(w) \right) \tag{59}$$

where

$$M(z) = \begin{cases} +1, & \text{for} & \psi_{1,3}; \\ -1, & \text{for} & \psi_{2,4}, \end{cases} \tag{60}$$

We see from (58,59) that condition $Q_1 = Q_2 = Q$ is insuring the full cancellation of $G(|z-w|)$ dependent parts in $Z_{1,2}$, giving

$$Z_1 = Z_2 = 1 \tag{61}$$

The term Z_3 in (56) is defined by the correlators of $V_{A,B}$ with φ_α dependent parts of ψ_i. It is easy to see, that

$$Z_3 \;=\; \sum_{\substack{\text{all possible} \\ \text{coverings}}} \prod_{\sigma=1}^{k} \prod_{\vec{n}\in\gamma_\sigma} \rho e^{\Gamma_{\vec{n},\vec{\mu}}} \prod_{z\in\gamma_\sigma} \bar{C}_i(z)K^+(z)K(z+\mu)C_j(z+\mu), \tag{62}$$

where

$$K(z) = e^{\frac{Q}{4\pi}\frac{Q}{2}\sum_w \Omega(w)[G(z-w)-G(\bar{z}-\bar{w})]}. \tag{63}$$

and the sum is taken over middle points of all (A and B) plaquettes.

The product $K^+(z)K(z+\mu)$ in (62), being element of $U(1)$ group, is nothing but $U(1)$ gauge field part in the action (1) in fixed gauge. The integration over fermionic

19

fields $C_i(z), \bar{C}_i(z)$ in partition function is simple and produces (-1) for the each closed contour Γ_σ in the (62) together with the product of expressions

$$K^+(z)K(z+\mu) = e^{\sum_w \frac{Q^2}{4\pi}\Theta_{z,z+\epsilon}(w)\Omega(w)}, \tag{64}$$

along them. In (64) $\Theta_{z,z+\mu}$ is the looking angle of the link $(z, z+\mu)$ from the middle point w of some plaquette in ML (see Fig.2) and is equal to 2π. Finally, by use of (52), we have the following expression for each contour in (62)

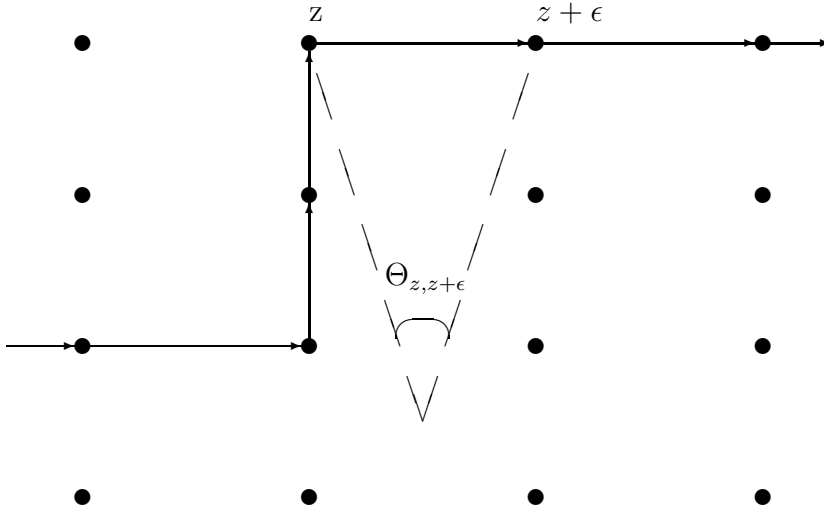$$\prod_{\text{along c}} K^+(z)K(z+\mu) = e^{i\pi Q^2}. \tag{65}$$

Fig.2.

In order to have a phase factor $e^{2\pi ip/q}$ which we need, one should take

$$Q^2 = 2p/q \tag{66}$$

One can obtain (65), taking into account (52) and the fact, that only points inside the contour C in (64) contribute to the phase and due to the Manhattan nature of the lattice we always have an extra $+$ or $-$ value for the $\Omega(w)$. An additional $-$ sign for each closed contours appeared because of the fermionic measure for the loops.

Therefore, after integration over φ_α, the whole partition function $Z(0)$ reduces to the product of factors

$$Z(0) = Z_\varphi \int \prod_{i=1}^{4} \prod_{\vec{n}} dC_{\vec{n}}^i d\bar{C}_{\vec{n}}^i \prod_{\substack{\text{all lattice} \\ \text{points}}} \bar{C}_i(z) C_i(z) \sum_{\text{all coverings}} \prod_{\sigma=1}^{k} (-e^{i\Phi_\sigma}) \prod_{\vec{n} \in \gamma_\sigma} \rho e^{\Gamma_{\vec{n},\vec{\mu}}}. \quad (67)$$

In the limit $\frac{m}{t} \to 0$ the last component of the product in (67) will be cancelled by the rescaling factor t^{-LN}. The $C, \bar{C}$-part and the last term of this expression represents the variety of coverings of the lattice by fermionic curves with the fixed number of loops and, as it was pointed out in the section 2, is equal to $Z(c = -2)$ in the limit of fully packed phase $\frac{m}{t} \to 0$. As for partition function of free scalars Z_φ in continuum limit, in $SO(2)$ rotation invariant regularization we have

$$Z_\varphi = \exp\left(-\frac{1}{48\pi^2} \int \Omega(z) G(|z - w|)\Omega(w)\right) \quad (68)$$

for each of them, which corresponds to central charge $c = 1$. Therefore, the fermionic and bosonic parts the expression (67) cancel each other and we are left with

$$Z(0) = \sum_{\text{all coverings}} \prod_{\sigma=1}^{k} (-e^{i\Phi_\sigma}), \quad (69)$$

where Φ_σ is the $U(1)$ flux inside the contour γ_σ and this expression indeed coincides with the (10) in the $\frac{m}{t} \to 0$ limit. Now it is clear why in the section 3 we have considered a particular scaling limit. Only in that unique limit we can have cancellation of bosonic and fermionic anomalous parts and our partition function will represent the fully packed limit of the model, defined in (10).

Expression (68) together with (58) shows that the central charges of scalars are $1 \pm 3Q^2$.

Finally let us make the following remark. Introduced two free scalars with the opposite statistics are equivalent to two dimensional vector field

$$A_\alpha = \partial_\alpha \varphi_1 + \epsilon_{\alpha\beta} \partial_\beta \varphi_2 \quad (70)$$

with the statistical weights equal

$$P(A_\alpha) = \exp\left(\int A_\alpha^2\right). \quad (71)$$

21

5   Continuum Limit of the Model

In the previous sections we have represented model as a free fermionic spin (0.1) system with central charge $c = -2$, interaction of which with the external magnetic field can be constructed via two scalars of opposite statistics and vacuum charge $Q^2 = 2p/q$. For the spin (0,1) fermionic system the action is

$$A(\psi_i, 0) = \frac{1}{\pi} \int \left(\bar{\psi}_L \partial \psi_L + \bar{\psi}_R \bar{\partial} \psi_R \right) \tag{72}$$

where $\psi_R(\psi_L)$ corresponds to $\psi_{2,4}(\psi_{1,3})$ fermions.

The formulation of continuum limit of the vacuum charge parts of the scalar fields in the action, which corresponds to background flux operator, is straightforward (see expressions of vertex operators in (54))

$$
\begin{aligned}
\prod_{z \in \text{plaquettes}} V_A(z, \bar{z}) V_B(z, \bar{z}) &= \exp \left(\sum_w i \frac{Q}{4\pi} \Omega(w)(\varphi_1 + \varphi_2) \right) = \\
&= \exp \left(i \frac{Q}{4\pi} \int d^2 w R(w)(\varphi_1 + \varphi_2) \right)
\end{aligned}
\tag{73}
$$

One can generalize the model (4) for the random surfaces. All closed random ML-s are in one to one correspondence with the closed surfaces in regular $3D$ lattice. Following [3] they can be constructed as follows. Let us consider middle points of the links on such surface in regular lattice as sites of the new lattice which we would like to construct and connect them with links. Now, by drawing arrows on these links as in Fig.3 we will complete the construction. The B_1 and B_2 plaquettes in Fig.3 coincides after rotation on $\pi/2$.
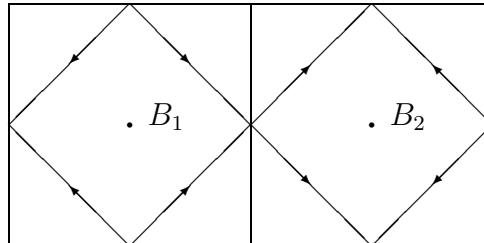

Fig.3.

The $B_{1,2}$ faces of ML are always quadrangles, while $A_{1,2}$ type faces, which formed around sites of the original $3D$ regular lattice, can be arbitrary n-angle, with $n = 3,4...9$ (in case of regular target lattice). Hence, the curvature of $A_{1,2}$ faces can be changed and become

$$\Omega_n = \frac{\pi}{2}(4 - n) \tag{74}$$

We would like to have a $U(1)$ flux in the n-angle face on random ML, equal to $\frac{\pi}{2}n$, which is proportional to area of the n-angle. Then the corresponding solid angle can be written as

$$\Omega(w) = \begin{cases} \Omega_0 - \Omega_n(w) = 2\pi - \Omega_n(w) & w \in A; \\ -\Omega_0 = -2\pi & w \in B, \end{cases} \tag{75}$$

and the formula for the loop gas representation of the models partition function (10) for random surface will be modified as follows

$$Z = \sum_{\substack{\text{all possible} \\ \text{coverings}}} \prod_{\sigma,\tau} (-\eta(\gamma_\sigma))(-\eta(\bar{\gamma}_\tau))\Phi(\gamma_\sigma)\Phi(\bar{\gamma}_\tau), \tag{76}$$

where

$$\eta(\gamma_\sigma) = exp\left(2\pi\frac{p}{q}\int_D \Omega\right), \tag{77}$$

with $\gamma_\sigma = \partial D$. So we see, that response of the model on inclusion of gravity is simply consists in appearance of the gravitational curvature of the random surface in the vacuum charge term (73) of the effective action S

$$R = R_{\text{background}} + R_{\text{grav}} \tag{78}$$

Finally we get the action of the defined model for the continuum limit as

$$\begin{aligned} S &= \frac{1}{\pi} \int d^2z \left[\bar{\psi}_L \partial \psi_L + \bar{\psi}_R \bar{\partial} \psi_R + \frac{1}{2}\partial\varphi_1\bar{\partial}\varphi_1 - \frac{1}{2}\partial\varphi_2\bar{\partial}\varphi_2 - \right. \\ &\quad \left. - i\frac{Q}{8}R(\varphi_1 + \varphi_2)\right]. \end{aligned} \tag{79}$$

Here the fermionic field ψ is the $(0,1)$ spin system with central charge equal to -2, which is necessary to cancel $+2$ part of the central charge of free fields φ_1 and φ_2, appearing in continuum limit in addition to $3Q^2$, calculated in (58).

It is easy to recognize in the obtained action the action for twisted $N = 2$ superconformal theories [24, 25], but in order to fix the concrete type of the theory, the $SU(2)/U(1)$ coupling constant k, it is necessary also to compare the $U(1)$ current. The usual $U(1)$ current on lattice is defined by hopping of fermions and the z (correspondingly $\bar{z}$) component of such current simply is

$$J_z = J_{\vec{n},\vec{\mu_1}} + iJ_{\vec{n}+\vec{\mu_1}+\vec{\mu_2},-\vec{\mu_2}} = \bar{C}^4_{\vec{n}+\vec{\mu_1}}(C^1_{\vec{n}} + iC^3_{\vec{n}+\vec{\mu_1}+\vec{\mu_2}}), \tag{80}$$

where μ_1 and μ_2 are the vectors on the ML in x and y directions correspondingly. By use of expressions (53) for $C^i(z,\bar{z})$, and with the help of properties of the vertex operators, it is not hard to find, that

$$J_z = \bar{\psi}_L\psi_L + iQ\partial\varphi_1. \tag{81}$$

This expression of $U(1)$ current, together with the action (79), allows to fix a model as twisted [8] $N = 2$ superconformal theory [24, 25] at the level

$$k + 2 = \frac{2}{Q^2} = \frac{q}{p} \tag{82}$$

The connection of N=2 superconformal theories with the $2d$ polymer physics, percolation problem and polymers at the Θ-point was investigated in [26, 27].

On the other side it is known [28, 29] that twisted $N = 2$ superconformal theories at the level k are equivalent to (p, q) minimal models, interacting with $2d$ gravity and are topological. So, the described model of $\pm 2\pi p/q$ fluxes can be considered as (p, q) minimal model interacting with $2d$-gravity and as microscopic statphysical equivalent for topological theories. We hope in this way to find a topological definition for sign factor of $3DIM$.

6   Acknowledgments

I appreciate very much the discussions with J.Ambjorn, H.Babujian, P.van Baal, T.Hakobyan, D.Karakhanyan, I.Kostov, H.Leutwyler and R.Poghossian. I am grateful to Newton Institute of Mathematical Science at Cambridge for hospitality, where the part of this work was done and Schweizerischer Nationalfond for support in part.

7 Appendix

Let us first calculate $Z(0)$, which simply connected with the determinant of the operator $\mathcal{H}(\vec{k})$, defined by eq. (5) as

$$
\begin{aligned}
Z(0) &= \prod_{\vec{k}} det[\mathcal{H}(\vec{k})] = \prod_{l,n} \{1 + (t_{14}t_{32} - t_{12}t_{34})(t_{23}t_{41} - t_{43}t_{21}) + \\
&+ \left(t_{23}t_{32}e^{2ik_x} + t_{14}t_{41}e^{-2ik_x} + t_{43}t_{34}e^{2ik_y} + t_{12}t_{21}e^{-2ik_y}\right)\},
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
k_x &= \frac{2\pi}{L}l, \quad l = 0,...\frac{L}{2} - 1, \\
k_y &= \frac{2\pi}{N}n, \quad n = 0,...\frac{N}{2} - 1.
\end{aligned}
\tag{84}
$$

The aim is to represent $Z(0)$ as

$$
Z(0) = TrT^{N/2} = \prod_{k_x} Tr(T_{k_x})^{N/2},
\tag{85}
$$

where

$$
T_{k_x} = \mathcal{D}\exp\left((\varepsilon_{k_x} + \mu)n_{1,k_x} + (-\varepsilon_{k_x} + \mu)n_{2,k_x}\right).
\tag{86}
$$

Then

$$
\begin{aligned}
Z(0) &= \prod_{k_x} \mathcal{D}^{\frac{N}{2}}\left(1 + e^{(\mu+\varepsilon_{k_x})\frac{N}{2}}\right)\left(1 + e^{(\mu-\varepsilon_{k_x})\frac{N}{2}}\right) \\
&= \prod_{k_x} 2\mathcal{D}^{\mu\frac{N}{2}}\left(\cosh\frac{N}{2}\mu + \cosh\frac{N}{2}\varepsilon_{k_x}\right).
\end{aligned}
\tag{87}
$$

Now, if we will use the known trigonometric identity

$$
\cosh\frac{N}{2}\varepsilon_{k_x} + \cosh\frac{N}{2}\mu = 2^{\frac{N}{2}-1}\prod_{n=0}^{\frac{N}{2}-1}\left\{\cosh\varepsilon_{k_x} + \cos(i\mu + \frac{2\pi}{N/2}n)\right\}
\tag{88}
$$

25

in the expression (87) for $Z(0)$, and compare it with the (83), we will find an exact coincidence, provided that

$$
\begin{aligned}
\mathcal{D} &= 2t_{12}t_{21} \\
e^{2\mu} &= \frac{t_{34}t_{43}}{t_{12}t_{21}} \\
\cosh\varepsilon_p &= \frac{1 + (t_{14}t_{32} - t_{12}t_{34})(t_{23}t_{41} - t_{43}t_{21})}{2(t_{12}t_{21}t_{34}t_{43})^{1/2}} + \\
&+ \frac{t_{23}t_{32}e^{2ip} + t_{14}t_{41}e^{-2ip}}{2(t_{12}t_{21}t_{34}t_{43})^{1/2}}.
\end{aligned}
\tag{89}
$$

$$\tag{90}$$

In the scaling limit of $Z(0)$, defined by eq.(37) and $\Phi \to 0$, we will have the same expression for the partition function as for Hamiltonian walk obtained in [17], where was shown, that it coincides with the partition function of $c = -2$ particles on the torus.

References

[1] A.Polyakov, Phys.Lett B82 (1979) 247

Vl.Dotsenko, A.Polyakov, in Advanced Studies in Pure Math. 15 (1987)

E.Fradkin, M.Srednicki, L.Susskind, Phys.Rev. D21(1980) 2885

S.Samuel, J.Math.Phys. 21 (1980) 2806, 2815, 2820

C.Itzikson, Nucl.Phys. B210(1982) 477

A.Cacher. D.Foerster, P.Winday, Nucl.Phys.B251 (1985) 29

[2] A.Sedrakyan, Phys.Lett. B137 (1984) 397

[3] A.Kavalov, A.Sedrakyan, Nucl.Phys. B285 [FS19](1987) 264

[4] Prange, S. Girvin, The Quantum Hall Effect (Springer, Berlin, 1990)

[5] R.B.Laughlin, Phys.Rev.Lett. 50 (1983) 1395

[6] B.Halperin, Phys.Rev. B 25 (1982) 2185

[7] X.-G.Wen, Phys.Rev. B 43 (1991) 11025

[8] T.Eguchi, S.K.Yang, Mod.Phye.Lett., A5, N21,(1990) 1693

[9] E.Verlinde, H.Verlinde, Nucl. Phys., B348 (1991) 457

[10] E.Witten, Commun.Math.Phys., 117 (1988) 353; Nucl.Phys. B340 (1990) 281

[11] S.Ouvry, R.Stora, P.Van Baal, Phys.Lett., B220 (1989) 159

[12] Whitney, Ann.of Math.45 (1949) 247
V.Arnold, A.Varchenko, S.Goussain-Zade, Singularites des applications differentiables (MIR, Moscow, 1986)

[13] J.Chalker, P.D.Coddington, J.Phys. C 21(1988) 2665

[14] article in preparation

[15] P.W.Kasteleyn, Physica, 29 (1963) 1329

[16] B.Nienhuis, Phys.Rev.Lett. 49 (1982) 1062
B.Nienhuis, J.Stat.Phys. 34 (1984) 731
B.Nienhuis, Phase Transition and Critical Phenomena,v.11,
eds. C.Domb, J.Lebowitz (Academic Press, London 1987)

[17] B.Duplantier, F.David, Journal of Stat.Phys. 51 (1988) 327

[18] B.Duplantier, Physica, D38 (1989) 71
B.Duplantier, Physics Reports C184,229-257

[19] B.Duplantier, I.Kostov, Nucl.Phys., B340 (1990) 491
I.Kostov, M.Staudacher, Nucl.Phys., B384 (1992) 459

[20] F. Berezin, The Methods of Second Quantization (Nauka,Moscow 1965) ,
L.Faddeev, Introduction to Functional Methods, in Les Houches 1975,
Session 28, Ed. R.Balian , J.Zinn-Justin

[21] A.Lerda, S.Sciuto, Nucl.Phys., B401 (1993) 613

[22] G.Mahan, Many Particle Physics (Plenum Press 1981) ,
     J.Solyom, Adv.Phys. 28 (1979) 201 ,
     F.D.M.Haldane, J.Phys. C14 (1981) 2585

[23] D.Friedan, S.Shenker, E.Martinec, Nucl.Phys. B271 (1986) 92

[24] G.Mussardo, G.Sotkov, M.Staniskov, Int.J.Mod.Phys., A4 (1989) 1135

[25] K.Ito, Phys.Lett., B230 (1989) 71

[26] H.Saleur, Nucl.Phys., B382 (1992) 486

[27] H.Saleur, Nucl.Phys., B382 (1992) 532

[28] S.Mukhi, C.Vafa, Harvard preprint HUTP-93/A002, hepth/9301083
     B.Gato-Riviera, A.Semikhatov, Phys.Lett., B293 (1992) 72
     M.Bershadsky, W.Lerche, D.Nemeschansky, N.Warner, Nucl.Phys., B401 (1993) 304

[29] N.Ohta, H.Suzuki, Mod.Phys.Lett. A9 (1994)541

[30] J.Cardy, Nucl.Phys., B270 [FS16](1986) 186
     H.W.J.Blote,J.Cardy and M.P.Nightingale, Phys.Rev.Lett., 56 (1986) 742
     I.Affleck, Phys.Rev.Lett. 56(1986) 746

[31] N.Hotano,D.Nelson, Phys.Rev.Lett. 77(1996) 570
     N.Hotano,D.Nelson, Phys.Rev. B 56(1997) 8651
     N.Shnerb, Phys.Rev. B 55 (1997) R3382
     J.Feinberg, A.Zee, Nucl.Phys. B504 (1997) 579
     E.Brezin, A.Zee,-Cond-matt/9708029

[32] K.Efetov, cond-matt/9703087

[33] L.W.Chen, L.Balentz, M.P.A.Fisher and M.C.Marchetti, Phys.Rev. B 54 (1996) 12798